Sign and area in nodal geometry of Laplace eigenfunctions

Fëdor Nazarov, Leonid Polterovich and Mikhail Sodin

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Abstract

The paper deals with asymptotic nodal geometry for the Laplace-Beltrami operator on closed surfaces. Given an eigenfunction f corresponding to a large eigenvalue, we study local asymmetry of the distribution of $\mathrm{sign}(f)$ with respect to the surface area. It is measured as follows: take any disc centered at the nodal line $\{f = 0\}$, and pick at random a point in this disc. What is the probability that the function assumes a positive value at the chosen point? We show that this quantity may decay logarithmically as the eigenvalue goes to infinity, but never faster than that. In other words, only a mild local asymmetry may appear. The proof combines methods due to Donnelly-Fefferman and Nadirashvili with a new result on harmonic functions in the unit disc.

1 Introduction and main results

Consider a compact manifold S endowed with a C^∞ Riemannian metric g. Let $\{f_\lambda\}$, $\lambda \nearrow +\infty$, be any sequence of eigenfunctions of the Laplace-Beltrami operator Δ_g:

$$\Delta_g f_\lambda + \lambda f_\lambda = 0\,.$$

The eigenfunctions f_λ give rise to an interesting geometric object, nodal sets $L_\lambda = \{f_\lambda = 0\}$. Each L_λ is a closed hypersurface with quite tame singularities. For instance, when S is 2-dimensional, any nodal line L_λ at a singular point p looks like the union of an even number of smooth rays meeting at p at equal angles [25, Chapter III]. In spite of this "infinitesimal

simplicity", the global picture of nodal sets for large λ becomes more and more complicated. This is partially due to the fact that L_λ is $\sim 1/\sqrt{\lambda}$-dense in S.

Asymptotic geometry of nodal sets as $\lambda \nearrow \infty$ attracted a lot of attention of both mathematicians and physicists though it is still far from being understood (see [16, 4] for discussion on recent developments). The idea of studying the asymptotic behavior comes from quantum mechanics where f_λ^2 (properly normalized) is interpreted as the probability density of the coordinate of a free particle in the pure state corresponding to f_λ, and $\lambda \nearrow +\infty$ corresponds to the quasi-classical limit.

A nodal domain is a connected component of the set $S \setminus L_\lambda$. All nodal domains can be naturally grouped into two subsets $S_+(\lambda) := \{f_\lambda > 0\}$ and $S_-(\lambda) := \{f_\lambda < 0\}$. Our story starts with two fundamental results obtained by Donnelly and Fefferman.

The first one is *a "local version" of the Courant nodal domain theorem* [10]: let $D \subset S$ be a metric ball and let U be any component of $S_+(\lambda) \cap D$ such that

$$U \cap \frac{1}{2}D \neq \varnothing\,, \tag{1.1}$$

which means that U enters deeply enough into D [1]. Then

$$\frac{\mathrm{Volume}(U)}{\mathrm{Volume}(D)} \geqslant a \cdot \lambda^{-k} \tag{1.2}$$

where a depends only on the metric g and k only on the dimension of S. The rate of decay of the right hand side (a negative power of λ) cannot be improved — a suitable example can be easily produced already in the case of standard spherical harmonics. The sharp value of the constant k is, however, still unknown (see papers [8, 22] for estimates on k).

The second result is the following *quasi-symmetry theorem* proved in [9, p. 182] under the extra assumption that the metric g is real analytic. Let $D \subset S$ be a fixed ball. Then there exists Λ depending on the radius of the ball D and the metric g such that for all $\lambda > \Lambda$

$$\frac{\mathrm{Volume}(S_+(\lambda) \cap D)}{\mathrm{Volume}(D)} \geqslant a, \tag{1.3}$$

[1] Here and below $\frac{1}{2}D$ stands for the ball with the same center as D whose radius equals half of the radius of D. The radii of all metric balls are assumed to be less than the injectivity radius of the metric.

where $a > 0$ depends only on the metric g.

From the geometric viewpoint, there is a significant difference between the measurements presented above: the quasi-symmetry theorem (1.3) deals with a ball of fixed radius and large λ. In contrast to this, the local version of the Courant theorem (1.2) is valid for all scales and all λ's though the collection of balls depends on λ through the "deepness assumption" (1.1). A natural problem arising from this discussion is to explore what remains of quasi-symmetry on all scales and for all λ, provided that the nodal set enters deeply enough into a ball: $L_\lambda \cap \frac{1}{2}D \neq \varnothing$.

In the present paper we deal with this problem in the case when S is a compact connected surface and the metric g is C^∞-smooth. Our main finding is that only a mild local asymmetry may appear. If in formula (1.2) one replaces a single component U by the whole set $S_+(\lambda)$, the right hand side changes its behavior: instead of a negative power of λ, it becomes $(\log\lambda)^{-1}(\log\log\lambda)^{-1/2}$. Moreover, we will show that even for the standard spherical metric it cannot be better than $(\log\lambda)^{-1}$. We believe that the double logarithm factor reflects a deficiency in our method (see discussion in Section 7.1). The precise formulations follow.

Theorem 1.4 *Let S be a compact connected surface endowed with a smooth Riemannian metric g, and let f_λ, $\lambda \geqslant 3$, be an eigenfunction of the Laplace-Beltrami operator. Assume that the set $S_+(\lambda) := \{f_\lambda > 0\}$ intersects a metric disc $\frac{1}{2}D$. Then*

$$\frac{\mathrm{Area}(S_+(\lambda)\cap D)}{\mathrm{Area}(D)} \geqslant \frac{a}{\log\lambda\cdot\sqrt{\log\log\lambda}}$$

where the constant $a > 0$ depends only on g.

(The condition $\lambda \geqslant 3$ is imposed here only because $\log$ and $\sqrt{\ }$ are not defined if the argument is less than 0. For $\lambda < 3$ the theorem holds with the right hand side replaced by a constant $a > 0$ depending on g only.)

The next result illustrates sharpness of the previous estimate up to the double logarithm [2]:

[2]It is worth mentioning that on "microscopic scales", when the radii of discs D are less than $(\lambda\log\log\lambda)^{-1/2}$, our approach gives an optimal bound

$$\frac{\mathrm{Area}(S_+(\lambda)\cap D)}{\mathrm{Area}(D)} \geqslant \frac{a}{\log\lambda}\,.$$

Theorem 1.5 *Consider the 2-sphere $\mathbb{S}^2$ endowed with the standard metric. There exist a positive numerical constant C, a sequence of Laplace-Beltrami eigenfunctions f_i, $i \in \mathbb{N}$ corresponding to eigenvalues $\lambda_i \to \infty$, and a sequence of discs $D_i \subset \mathbb{S}^2$ such that each f_i vanishes at the center of D_i and*

$$\frac{\mathrm{Area}(S_+(\lambda_i) \cap D_i)}{\mathrm{Area}(D_i)} \leqslant \frac{C}{\log \lambda_i}\,.$$

After Donnelli and Fefferman [9], various versions of quasi-symmetry for eigenfunctions were studied by Nadirashvili [23] and Jakobson-Nadirashvili [15]. To a high extent, the present research was stimulated by Nadirashvili's article [23].

Our approach to Theorem 1.4 is based on the analysis of the eigenfunctions f_λ on discs of radius $\sim 1/\sqrt{\lambda}$. The proof consists of four main ingredients that we are going to describe right now. The first three of them exist in the literature. Our innovation is the last one, namely, the calculation of the asymptotical behaviour of the Nadirashvili constant for harmonic functions.

Donnelly-Fefferman growth bound. For any continuous function f on a closed disc D (in any metric space), define its *doubling exponent* $\beta(D, f)$ by

$$\beta(D, f) = \log \frac{\max_D |f|}{\max_{\frac{1}{2}D} |f|}.$$

The following fundamental inequality was established in [9] in any dimension. For any metric disc $D \subset S$ and any λ,

$$\beta(D, f_\lambda) \leqslant a\sqrt{\lambda} \tag{1.6}$$

where the constant a depends only on the metric g.

Reduction to harmonic functions. Assume now that $D \subset S$ is a disc of radius $\sim 1/\sqrt{\lambda}$. It turns out that on this scale the eigenfunction f_λ can be "approximated" by a harmonic function u on the unit disc $\mathbb{D}$. More precisely, the set $\{f_\lambda > 0\}$ can be transformed into the set $\{u > 0\}$ by a K-quasiconformal homeomorphism with a controlled dilation K. Moreover, the doubling exponent of u on $\mathbb{D}$ is essentially the same as that of f_λ in D. This idea is borrowed from Nadirashvili's paper [23]. The details are presented in Section 3 below.

Topological interpretation of the doubling exponent. Let $u\colon \mathbb{D}\to\mathbb{R}$ be a non-zero harmonic function. Denote by $\nu(r\mathbb{T},u)$ the number of sign changes of u on the circle $r\mathbb{T}=\{|z|=r\}$. Then

$$C^{-1}(\beta(\tfrac14\mathbb{D},u)-1)\leqslant \nu(\tfrac12\mathbb{T},u)\leqslant C(\beta(\mathbb{D},u)+1) \tag{1.7}$$

where C is a positive numerical constant. This result goes back to Gelfond [11] (cf. [24], [14], and [18, Theorem 3]). We will need the inequality on the right only, which will be proved in Section 2. The inequality on the left is presented here just for completeness.

The Nadirashvili constant. Denote by $\mathcal{H}_d$ the class of all non-zero harmonic functions u on $\mathbb{D}$ with $u(0)=0$ that have no more than d sign changes on the unit circle $\mathbb{T}$. Define the *Nadirashvili constant*

$$\mathcal{N}_d := \inf_{u\in\mathcal{H}_d} \operatorname{Area}(\{u>0\})\,.$$

Using an ingenious compactness argument, Nadirashvili [23] showed that $\mathcal{N}_d$ is strictly positive. Our next result gives a satisfactory estimate of the Nadirashvili constant:

Theorem 1.8 *There exists a positive numerical constant C such that for each $d\geqslant 2$,*

$$\frac{C^{-1}}{\log d}\leqslant \mathcal{N}_d\leqslant \frac{C}{\log d}\,.$$

Nadirashvili's proof of positivity of $\mathcal{N}_d$ is non-constructive, hence we had to take a different route. Our approach is based on one-dimensional complex analysis.

The four steps described above yield Theorem 1.4 in the case when the disc D is *small*, that is, of radius $\leqslant a\lambda^{-1/2}$. The double logarithm term is the price we pay for the fact that the transition from the eigenfunction f_λ to the approximating harmonic function u is given by a quasiconformal homeomorphism, which in general is only Hölder. The case of an arbitrary (not necessarily small) disc D is based on the following standard argument. The nodal line $L=\{f_\lambda=0\}$ is $\sim 1/\sqrt{\lambda}$-dense in S (see e.g. [6]). Hence every disc D with $L\cap\frac12 D\neq\varnothing$ contains a disjoint union of small discs D_i whose centers lie on L and such that the total area of these discs is $\geqslant \mathrm{const}\cdot\operatorname{Area}(D)$. Since the area bound is already established for each D_i, it extends with a

weaker constant to D. This completes the outline of the proof of Theorem 1.4.

Organization of the paper. The next section is devoted to harmonic functions on the unit disc. We establish the lower bound $\mathcal{N}_d \geqslant c(\log d)^{-1}$ for the Nadirashvili constant and prove the right inequality in (1.7) relating the number of boundary sign changes to the doubling exponent.

In Section 3, we deal with solutions of the Schrödinger equation in the unit disc with small potential. This Schrödinger equation is nothing else but an appropriately rescaled equation $\Delta_g f + \lambda f = 0$ written in local conformal coordinates on the surface. For the solutions F of this equation, we prove a lower bound on Area($\{F > 0\}$ in terms of the doubling exponent of F. The proof is based on a quasiconformal change of variables that reduces the problem to the estimate for harmonic functions obtained in Section 2.

In Section 4, we present the easiest proof of the Donnelli-Fefferman fundamental inequality (1.6) we are aware of. We should warn the reader that our proof works in dimension 2 only. It is based on a simple observation about second order linear ODEs in Hilbert spaces. The reader familiar with the Donnelli-Fefferman inequality [9, 17] can disregard this section.

At this point we have all the ingredients necessary to prove Theorem 1.4. This is done in Section 5.

In Section 6, we present examples illustrating the local logarithmic asymmetry for harmonic functions and Laplace-Beltrami eigenfunctions. Our construction uses the complex double exponential function $\exp\exp z$. We confirm the upper bound for the Nadirashvili constant $\mathcal{N}_d$, which completes the proof of Theorem 1.8. Then, "transplanting" the Taylor series of the obtained harmonic function at 0 to the north pole of the unit sphere, we obtain Theorem 1.5 that shows that our main result is already sharp for spherical harmonics up to the double logarithm.

The paper concludes with discussion and questions. In particular, we indicate a link between the expectation of the doubling exponent of an eigenfunction f_λ on a random disc of radius $\sim 1/\sqrt{\lambda}$ and the length of its nodal line $\{f_\lambda = 0\}$.

Convention. Throughout the paper, we denote by c, c_0, c_1, c_2, ... positive numerical constants, and by a, a_0, a_1, ... positive constants that depend only on the metric g. In each section we start a new enumeration of these constants.

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2 The area estimate for harmonic functions

In this section, we show that for any non-zero harmonic function u on $\mathbb{D}$ vanishing at the origin,

$$\mathrm{Area}(\{u>0\}) \geqslant \frac{c}{\log \nu(\mathbb{T},u)}\,, \tag{2.1}$$

i.e., we prove the lower bound for the Nadirashvili constant $\mathcal{N}_d$ in Theorem 1.8. Then we prove the right hand part of estimate (1.7). Together with (2.1), it yields

Theorem 2.2 *Let u be a non-zero harmonic function on the unit disc $\mathbb{D}$ vanishing at the origin. Then*

$$\mathrm{Area}(\{u>0\}) \geqslant \frac{c_0}{\log \beta^*(\mathbb{D},u)}$$

where $\beta^* := \max(\beta, 3)$.

Consider the analytic function $f\colon \mathbb{D}\to\mathbb{C}$ with $\mathrm{Re}\, f = u$ and $f(0)=0$. Assume that f does not vanish on $r\mathbb{T}$. Consider all arcs $L\subset r\mathbb{T}$ travelled counterclockwise (including the entire circumference $r\mathbb{T}$ viewed as an arc whose end and beginning coincide). Put

$$\omega(r\mathbb{T},f) := \max_{L\subset r\mathbb{T}} \Delta_L \mathrm{arg} f$$

where $\Delta_L \mathrm{arg} f$ is the increment of the argument of f over L, that is,

$$\Delta_L \mathrm{arg} f = \mathrm{arg} f(\theta_2) - \mathrm{arg} f(\theta_1)\,,$$

for $L=[\theta_1;\theta_2]$. We shall prove

Theorem 2.3 *Let f be an analytic function on $\mathbb{D}$ vanishing at the origin. Assume that $f|_{\mathbb{T}} \neq 0$. Then*

$$\mathrm{Area}(\{\mathrm{Re}\, f>0\}) \geqslant \frac{c_1}{\log \omega(\mathbb{T},f)}\,. \tag{2.4}$$

Since $\omega(\mathbb{T}, f) \leqslant \pi(\nu(\mathbb{T}, \mathrm{Re} f)+1)$, this yields estimate (2.1) and, therefore, the lower bound for the Nadirashvili's constant.

Proof of Theorem 2.3: For $k \in \mathbb{N}$, denote by $\mathcal{F}_k$ the class of analytic functions f on $\mathbb{D}$ such that $f(0) = 0$, f does not vanish on $\mathbb{T}$, and $\omega(\mathbb{T}, f) \leqslant 2\pi \cdot 2^k$. Put

$$A_k = \inf_{f \in \mathcal{F}_k} \mathrm{Area}(\{\mathrm{Re}\, f > 0\}) .$$

The estimate (2.4) would follow from the inequality

$$A_k \geqslant \frac{c_2}{k}. \tag{2.5}$$

Start with any $f \in \mathcal{F}_k$, and define δ by

$$1 - 2\delta = \sup\{r\colon\, f|_{r\mathbb{T}} \neq 0,\ \omega(r\mathbb{T}, f) < 2\pi \cdot 2^{k-1}\} .$$

If this set is empty, we simply take $\delta = \frac{1}{2}$.

Consider the annulus $E = \{1 - 2\delta < |z| < 1 - \delta\}$ and its subset $E_+ = \{z \in E\colon\ \mathrm{Re}\, f(z) > 0\}$. The heart of our argument is the following

Lemma 2.6 $\mathrm{Area}(E_+) \geqslant c_3\delta^2$.

Assuming the lemma, let us prove inequality (2.5) by induction on k. First of all, consider the case $k = 1$. Since f vanishes at the origin,

$$\omega(r\mathbb{T}, f) \geqslant \Delta_{r\mathbb{T}} \mathrm{arg} f \geqslant 2\pi$$

for all $r > 0$. Therefore, we can take $\delta = \frac{1}{2}$, and Lemma 2.6 yields $A_1 \geqslant \frac{1}{4}c_3$. Hence, taking $c_2 = \frac{1}{4}c_3$, we prove the induction base for claim (2.5).

Assume now that (2.5) is true for $k - 1$. Let us prove it for k. Take any $f \in \mathcal{F}_k$. If $\delta = \frac{1}{2}$, Lemma 2.6 immediately yields

$$\mathrm{Area}(\{\mathrm{Re}\, f > 0\}) \geqslant \mathrm{Area}(E_+) \geqslant c_2 \geqslant \frac{c_2}{k} .$$

Otherwise, we can find $r > 0$ arbitrarily close to $1 - 2\delta$ and such that f does not vanish on $r\mathbb{T}$ and $\omega(r\mathbb{T}, f) < 2\pi \cdot 2^{k-1}$. Put $g(z) = f(rz)$, $z \in \mathbb{D}$. Note that $g \in \mathcal{F}_{k-1}$ due to our choice of r. Obviously,

$$\mathrm{Area}(\{\mathrm{Re}\, f > 0\}) \geqslant \mathrm{Area}(E_+) + r^2 \mathrm{Area}(\{\mathrm{Re}\, g > 0\}) .$$

Applying Lemma 2.6 and the induction assumption and letting $r \to 1 - 2\delta$, we get

$$A_k \geqslant c_3\delta^2 + (1-2\delta)^2 \frac{c_2}{k-1} = \frac{c_3}{4}(2\delta)^2 + (1-2\delta)^2 \frac{c_2}{k-1} \,.$$

Note that the minimal value of the function $q(x) = \alpha x^2 + \beta(1-x)^2$ equals $\alpha \cdot \beta/(\alpha+\beta)$. Thus,

$$A_k \geqslant \frac{(c_3/4)\cdot(c_2/(k-1))}{c_3/4 + c_2/(k-1)} = \frac{c_2}{k + 4c_2/c_3 - 1} \,.$$

Hence, making the same choice $c_2 = c_3/4$ as above, we get $A_k \geqslant c_2/k$, and inequality (2.5) follows. This yields Theorem 2.3 modulo Lemma 2.6. $\square$

Proof of Lemma 2.6: The proof is based on comparing the upper and the lower bounds for the integral

$$\iint_{E_+} |\nabla \mathrm{arg} f| \, d\,\mathrm{Area} \,.$$

Any function $f \in \mathcal{F}_k$ admits the factorization

$$f(z) = e^{g(z)} \prod_{\zeta \in \mathcal{N}(f)} (z - \zeta)$$

where $\mathcal{N}(f)$ is the set of zeroes of f in $\mathbb{D}$ counted with their multiplicities and g is an analytic function in $\mathbb{D}$. Put $M := 2^k \cdot 2\pi$. Applying the argument principle, we conclude that the number N of zeroes of f in $\mathbb{D}$ satisfies

$$N \leqslant \frac{M}{2\pi} \,. \tag{2.7}$$

Further, for $|\zeta| < 1$, the function $\theta \to \arg(e^{i\theta} - \zeta)$ increases with θ. Therefore, considering the arc $L \subset \mathbb{T}$ joining the point of the minimum of $\mathrm{Im}\, g$ to the point of the maximum of $\mathrm{Im}\, g$ counterclockwise, we obtain

$$\operatorname*{osc}_{\mathbb{T}} \mathrm{Im}\, g := \max_{\mathbb{T}} \mathrm{Im}\, g - \min_{\mathbb{T}} \mathrm{Im}\, g \leqslant \Delta_L \mathrm{arg} f \leqslant M. \tag{2.8}$$

Fix $r \in (1-2\delta, 1-\delta)$ such that $r\mathbb{T} \cap \mathcal{N}(f) = \varnothing$. We call an open arc $I \subset r\mathbb{T}$ *a traversing arc* if its image curve $f(I)$ traverses the right half-plane, that is, a continuous branch of $\mathrm{arg} f$ maps I onto an interval $J = (-\frac{\pi}{2} + 2\pi m; \frac{\pi}{2} + 2\pi m)$

for some $m \in \mathbb{Z}$. Each traversing arc lies in the set E_+ which we are studying. By our choice of δ, the increment of the argument of f over some arc $L \subset r\mathbb{T}$ is at least $M/2$. Hence $L \cap E_+$ (and, thereby, $r\mathbb{T} \cap E_+$) contains either at least $M/(4\pi)$ pairwise disjoint traversing arcs or $\frac{M}{4\pi} - 1$ traversing arcs and two "tails". These tails, taken together, are as good for our purposes as one full traversing arc.

Given a traversing arc $I \subset r\mathbb{T}$, note that

$$\int_I |\nabla \mathrm{arg} f(z)|\, |dz| \geqslant \pi\, .$$

Summing up these inequalities over all traversing arcs lying on $r\mathbb{T}$ and integrating over $r \in (1-2\delta; 1-\delta)$, we get

$$\iint_{E_+} |\nabla \mathrm{arg} f|\, d\,\mathrm{Area} \geqslant \frac{M\delta}{4}\, . \tag{2.9}$$

On the other hand,

$$|\nabla \mathrm{arg} f(z)| \leqslant |\nabla\, \mathrm{Im}\, g(z)| + \sum_{\zeta \in \mathcal{N}(f)} \frac{1}{|z-\zeta|}\, . \tag{2.10}$$

Next, we use an estimate for the gradient of a harmonic function v in a disc D of radius t centered at c:

$$|\nabla v(c)| \leqslant \frac{2}{t} \max_{\partial D} |v|\, ,$$

which easily follows by differentiation of the Poisson integral representation for v in D. Applying this estimate to $v = \mathrm{Im}\, g - m$ with $m = \frac{1}{2}(\max_{\mathbb{T}} \mathrm{Im} g + \min_{\mathbb{T}} \mathrm{Im} g)$ and taking into account inequality (2.8), we readily get that

$$|\nabla\, \mathrm{Im}\, g| \leqslant \delta^{-1} \operatorname*{osc}_{\mathbb{T}} \mathrm{Im}\, g \leqslant \frac{M}{\delta} \tag{2.11}$$

everywhere in the annulus E. Further,

$$\begin{aligned} \iint_{E_+} \frac{d\,\mathrm{Area}(z)}{|z-\zeta|} &= \iint_{\zeta + E_+} \frac{d\,\mathrm{Area}(w)}{|w|} \\ &\leqslant \iint\limits_{|w| \leqslant \sqrt{\mathrm{Area}(E_+)/\pi}} \frac{d\,\mathrm{Area}(w)}{|w|} \leqslant 2\sqrt{\pi\, \mathrm{Area}(E_+)}\, . \end{aligned} \tag{2.12}$$

Estimates (2.10), (2.11), (2.12), and (2.7) give us

$$\iint_{E_+} |\nabla \arg f| \, d\,\mathrm{Area} \leqslant \frac{M}{\delta} \cdot \mathrm{Area}(E_+) + \frac{M}{2\pi} \cdot 2\sqrt{\pi \mathrm{Area}(E_+)}\,.$$

Juxtaposing this with (2.9) and canceling the factor M, we get

$$\frac{\delta}{4} \leqslant \frac{\mathrm{Area}(E_+)}{\delta} + \sqrt{\mathrm{Area}(E_+)/\pi}\,.$$

This yields $\mathrm{Area}(E_+) \geqslant c_3 \cdot \delta^2$, proving the lemma. □

In order to get Theorem 2.2, we need the following

Lemma 2.13 *Let u be a non-zero harmonic function on $\mathbb{D}$ vanishing at the origin. Then $\nu(\frac{1}{2}\mathbb{T}, u) \leqslant c_4 \beta^*(\mathbb{D}, u)$.*

Proof of Lemma 2.13: The proof is a minor variation of the argument used in [11, 14]. Consider the function

$$U(\theta) = u(\tfrac{1}{2}e^{i\theta}) = \sum_{k\in\mathbb{Z}} \widehat{u}(k) 2^{-|k|} e^{ik\theta}\,, \tag{2.14}$$

where $\{\widehat{u}(k)\}$ are the Fourier coefficients of the function $\theta \mapsto u(e^{i\theta})$. Since $|\widehat{u}(k)| \leqslant \max_{\mathbb{D}} |u|$, we see by inspection of formula (2.14) that the function U has an analytic extension onto the strip $\Pi = \{|\operatorname{Im}\theta| \leqslant \log\sqrt{2}\}$ and that

$$\max_{\theta\in\Pi} U(\theta) \leqslant \left(\sum_{k\in\mathbb{Z}} 2^{-|k|/2}\right) \cdot \max_{\mathbb{D}} |u| = c_5 \cdot \max_{\mathbb{D}} |u|\,. \tag{2.15}$$

At the same time,

$$\max_{\theta\in\mathbb{R}} |U(\theta)| = \max_{0.5\mathbb{D}} |u| = e^{-\beta(\mathbb{D},u)} \cdot \max_{\mathbb{D}} |u|\,. \tag{2.16}$$

Now observe that $\nu(1/2\mathbb{T}, u)$ does not exceed the number of zeroes of U on the interval $[-\pi, \pi]$. The latter can be easily estimated using Jensen's formula.

For this purpose, consider the rectangle $P = \{|x| \leqslant \frac{3\pi}{2}, |y| \leqslant \log\sqrt{2}\}$ and a conformal mapping $h : \mathbb{D} \to P$ with $h(0)$ chosen in such a way that

$$\max_{[-\pi,\pi]} |U| = |(U \circ h)(0)|\,.$$

There exists $c_6 < 1$ such that, for all such mappings h, we have $h^{-1}[-\pi, \pi] \subseteq \{|z| \leqslant c_6\}$. Denote by $n(t)$ the number of zeroes of the analytic function $U \circ h$ in the closed disc $\{|z| \leqslant t\}$. Then

$$\begin{aligned}
\nu(1/2\mathbb{T}, u) \quad &\leqslant \quad n(c_6) \leqslant \frac{1}{\log(1/c_6)} \int_{c_6}^{1} \frac{n(t)}{t}\, dt \leqslant \frac{1}{\log(1/c_6)} \int_{0}^{1} \frac{n(t)}{t}\, dt \\
&\overset{\text{'Jensen'}}{=} \quad \frac{1}{\log(1/c_6)} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |(U \circ h)(e^{i\theta})| d\theta - \log |(U \circ h)(0)| \right) \\
&\leqslant \quad \frac{1}{\log(1/c_6)} \left(\log \max_{\Pi} |U| - \log \max_{\mathbb{R}} |U| \right) .
\end{aligned}$$

Now, taking into account estimates (2.15) and (2.16), we readily see that the right hand side is $\leqslant (\log(1/c_6))^{-1}(\log c_5 + \beta(\mathbb{D}, u))$, proving the lemma. $\square$

3 An area estimate for solutions to Schrödinger's equation with small potential

In local conformal coordinates on the surface S, the equation $\Delta_g f_\lambda + \lambda f_\lambda = 0$ reduces to $\Delta f + \lambda q f = 0$. If the size of the local chart is comparable to the wavelength $\lambda^{-1/2}$, then, after rescaling and absorbing the spectral parameter λ into the potential q, one arrives at the Schrödinger equation

$$\Delta F + qF = 0 \tag{3.1}$$

with a bounded smooth potential q on the unit disc $\mathbb{D}$. The disc is endowed with the complex coordinate $z = x + iy$.

Throughout this section, we assume that $\|q\| := \max_{\mathbb{D}} |q| < \varepsilon_0$ where ε_0 is a sufficiently small positive numerical constant. The result of the present section is an intermediate step between Theorems 2.2 and 1.4.

Theorem 3.2 *Let F be any non-zero solution of equation* (3.1) *with* $F(0) = 0$. *Set*

$$\beta(F) := \sup_{D \subset \mathbb{D}} \beta(D, F)$$

(the supremum is taken over all discs $D \subset \mathbb{D}$). Set $\beta^*(F) = \max(\beta(F), 3)$. *Then*

$$\operatorname{Area}(\{F > 0\}) \geqslant \frac{c}{\log \beta^*(F) \cdot \sqrt{\log \log \beta^*(F)}}.$$

The proof is based on Theorem 2.2 and on a chain of lemmas. By $\|\cdot\|$ we always mean the uniform norm in $\mathbb{D}$.

Lemma 3.3 *If ε_0 is sufficiently small, then equation* (3.1) *admits a positive solution φ with*

$$1 - c_1\|q\| \leqslant \varphi \leqslant 1\,.$$

Proof of Lemma 3.3: Define recursively a sequence of functions F_i by $F_0 = 1$; $\Delta F_{i+1} = -qF_i$, $F_{i+1}|_{\mathbb{T}} = 0$. Then F_{i+1} can be represented in $\mathbb{D}$ as Green's potential of the function qF_i:

$$F_{i+1}(z) = \iint_{\mathbb{D}} \log\left|\frac{1 - z\bar{w}}{z - w}\right| q(w)F_i(w)\, d\,\mathrm{Area}(w)\,,$$

which readily yields $\|F_{i+1}\| \leqslant c_0\|q\|\,\|F_i\|$. Choosing $\varepsilon_0 < \frac{1}{2c_0}$, we get $\|F_i\| \leqslant (c_0\|q\|)^i \leqslant 2^{-i}$. Hence the series

$$\psi = \sum_{i=0}^{\infty} F_i$$

converges uniformly. Therefore ψ is a weak and thus a classical solution of the equation (3.1). Also,

$$\|\psi - 1\| \leqslant \sum_{i\geqslant 1} \|F_i\| \leqslant \frac{c_0\|q\|}{1 - c_0\|q\|} \leqslant 2c_0\|q\|\,.$$

Finally,

$$\varphi = \frac{\psi}{\|\psi\|}$$

is the desired positive solution. □

Lemma 3.4 *Let F be any non-zero solution to equation* (3.1)*. Then there exist a K-quasiconformal homeomorphism $h\colon \mathbb{D} \to \mathbb{D}$ with $h(0) = 0$ and a harmonic function $U\colon \mathbb{D} \to \mathbb{R}$ such that $F = \varphi \cdot (U \circ h)$. Moreover, the dilation K satisfies*

$$K \leqslant 1 + c_2\|q\|\,. \tag{3.5}$$

Proof of Lemma 3.4: Write $F = \varphi u$, and note that (by direct computation) equation (3.1) yields

$$\frac{\partial}{\partial x}\left(\varphi^2 \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varphi^2 \frac{\partial u}{\partial y}\right) = 0\,. \tag{3.6}$$

Thus, there exists a unique smooth function v with $v(0) = 0$ such that $\varphi^2 u_x = v_y$, and $\varphi^2 u_y = -v_x$. To rewrite these equations in the complex form, we consider the complex-valued function $w = u + iv$. An inspection shows that

$$\frac{\partial w}{\partial \bar{z}} = \frac{1-\varphi^2}{1+\varphi^2}\,\overline{\frac{\partial w}{\partial z}}\,.$$

In other words, w satisfies the Beltrami equation

$$\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z}$$

with the Beltrami coefficient

$$\mu = \frac{1-\varphi^2}{1+\varphi^2} \cdot \frac{u_x + iu_y}{u_x - iu_y}\,.$$

Clearly,

$$|\mu| = \frac{1-\varphi^2}{1+\varphi^2} < 1\,.$$

Since u is a non-trivial solution of an elliptic equation (3.6), its critical points are isolated. Thus, μ is a measurable function defined almost everywhere. By the fundamental existence theorem [2, Chapter V], there exists a K-quasiconformal homeomorphism $h\colon \mathbb{D} \to \mathbb{D}$ with $h(0) = 0$ such that $w = W \circ h$ where W is an analytic function on $\mathbb{D}$. This yields $F = \varphi \cdot (U \circ h)$ where $U = \operatorname{Re} W$. The dilation K of h satisfies

$$\frac{K-1}{K+1} \leqslant \|\mu\|_{L^\infty}\,.$$

Taking into account Lemma 3.3, we get inequality (3.5). □

The dilation K controls geometric properties of the homeomorphism h. We shall use Mori's theorem, which states that h is $\frac{1}{K}$-Hölder and

$$\frac{1}{16}|z_1 - z_2|^K \leqslant |h(z_1) - h(z_2)| \leqslant 16|z_1 - z_2|^{1/K} \tag{3.7}$$

(see [2, Section IIIC]), and Astala's distortion theorem [3]:

$$\operatorname{Area}(h(E)) \leqslant c_3 \operatorname{Area}(E)^{1/K} . \tag{3.8}$$

The constant c_3 in Astala's theorem depends on K but stays bounded when K remains bounded, so we may treat it as absolute.

Lemma 3.9 *We have*

$$\operatorname{Area}(\{F > 0\}) \geqslant \frac{c_4}{(\log \beta^*(F))^{1+c_2\|q\|}} . \tag{3.10}$$

Later we will show how this estimate can be improved by simple rescaling.

Proof of Lemma 3.9: We have $\{F > 0\} = h^{-1}\{U > 0\}$ where U is the harmonic function obtained in the previous lemma. Hence, by the area distortion theorem (3.8),

$$\operatorname{Area}(\{F > 0\}) = \operatorname{Area}\left(\{h^{-1}\{U > 0\}\}\right) \geqslant c_5 \operatorname{Area}(\{U > 0\})^K .$$

By Theorem 2.2,

$$\operatorname{Area}(\{U > 0\}) \geqslant \frac{c_6}{\log \beta^*(U, \mathbb{D})} .$$

Now, using Mori's theorem, we choose a positive integer ℓ_0 so large that $h^{-1}(\frac{1}{2}\mathbb{D}) \supset 2^{-\ell_0}\mathbb{D}$. Then

$$\frac{\max_{\mathbb{D}} |U|}{\max_{\frac{1}{2}\mathbb{D}} |U|} = \frac{\max_{\mathbb{D}} |U \circ h|}{\max_{h^{-1}(\frac{1}{2}\mathbb{D})} |U \circ h|} \leqslant \frac{\max_{\mathbb{D}} |U \circ h|}{\max_{2^{-\ell_0}\mathbb{D}} |U \circ h|} .$$

The right hand side is bounded by

$$c_7 \frac{\max_{\mathbb{D}} |F|}{\max_{2^{-\ell_0}\mathbb{D}} |F|} \leqslant c_7 e^{\ell_0 \beta(F)} .$$

Hence $\beta^*(U, \mathbb{D}) \leqslant c_8 \beta^*(F)$, and

$$\operatorname{Area}(\{F > 0\}) \geqslant \frac{c_9}{(\log \beta^*(F))^K} .$$

Recalling estimate (3.5), we get the desired result. □

The end of the proof of Theorem 3.2: The nodal set $L = \{F = 0\}$ of the function F contains the origin and does not have closed loops (since it is

homeomorphic to the nodal set of the harmonic function U). Therefore, for any $r \in (0,1]$, there are at least $c_{10} r^{-1}$ disjoint discs $D_j \subset \mathbb{D}$ of radius r centered at $z_j \in L$. For each disc D_j, consider the function

$$F_j(z) = F(z_j + rz)\,, \qquad z \in \mathbb{D}\,.$$

It satisfies the equation

$$\Delta F_j + q_j F_j = 0 \tag{3.11}$$

with $q_j(z) = r^2 q(z_j + rz)$, $\|q_j\| \leqslant r^2 \|q\| \leqslant r^2 \varepsilon_0$. Applying Lemma 3.9 to F_j instead of F and taking into account that $\beta^*(F_j) \leqslant \beta^*(F)$, we get

$$\mathrm{Area}(\{F_j > 0\}) \geqslant \frac{c_4}{(\log \beta^*(F))^{1+c_2\varepsilon_0 r^2}}\,.$$

To simplify the notation, denote $b = \log \beta^*(F)$ and $s = c_2 \varepsilon_0 r^2$, so that $\mathrm{Area}(\{F_j > 0\}) \geqslant c_4 b^{-1-s}$. Then

$$\begin{aligned}
\mathrm{Area}(\{F > 0\}) &\geqslant \sum_j \mathrm{Area}(\{F > 0\} \cap D_j) \\
&= r^2 \sum_j \mathrm{Area}(\{F_j > 0\}) \\
&\geqslant r^2 \cdot \frac{c_{10}}{r} \cdot \frac{c_4}{b^{1+s}} = \frac{c_{11}\sqrt{s}}{b^{1+s}}\,.
\end{aligned}$$

The choice of the scaling parameter r (and hence of s) is in our hands. One readily checks that, for $\beta \geqslant \log 3$, the function $s \mapsto \sqrt{s} b^{-s}$, $s \in (0, c_2\varepsilon_0]$, attains its maximum at $s = (2\log b)^{-1}$ for large b and at $s = c_2\varepsilon_0$ for small b. In both cases the maximal value of this function is $\geqslant c_{12} (\log b)^{-1/2}$. This completes the proof. $\square$

4 The Donnelly-Fefferman estimate

In this section, we prove the Donnelly-Fefferman estimate for the doubling exponent:

Theorem 4.1 *For any metric disc $D \subset S$,*

$$\beta(D, f_\lambda) \leqslant a_1 \sqrt{\lambda}\,.$$

Our proof is based on a version of the 'Three Circles Theorem' for solutions of the Schrödinger equation. Similar results are known under various assumptions: see Landis [20], Agmon [1], Gerasimov [12], Brummelhuis [5], Kukavica [19].

Let F be a solution to the equation

$$\Delta F + qF = 0 \tag{4.2}$$

where q is a smooth function in the unit disc $\mathbb{D}$. We no longer assume that q is small. Instead, the size of q will be controlled by the quantity

$$N = \max_{\mathbb{D}} \left(|q| + \rho |q_\rho| \right)$$

where ρ is the polar radius. Denote $M(r) = \max_{r\mathbb{D}} |F|$.

Theorem 4.3 *Let F be a solution to the equation* (4.2)*. Then*

$$\frac{M(2s)}{M(s)} \leqslant c_1 e^{c_2\sqrt{N}} \frac{M(8r)}{M(r)}\,, \tag{4.4}$$

provided that $0 < s \leqslant r \leqslant \frac{1}{8}$.

4.1 Proof of Theorem 4.3

Our first aim will be to replace PDE (4.2) by a second order ODE $\ddot{h} = L(t)h$ where $L(t)$ is a non-negative unbounded operator on a Hilbert space such that $\dot{L}(t)$ is also non-negative.

First, adding an extra variable z, we make the potential non-positive. Put $v(x,y,z) = F(x,y) \cdot \cosh \gamma z$ where $\gamma = \sqrt{N}$. Then $\Delta v = (\gamma^2 - q)v$, or, which is the same,

$$v_{rr} + \frac{2}{r} v_r = -\frac{1}{r^2} \widetilde{\Delta} v + (\gamma^2 - q) v \tag{4.5}$$

where r is the polar radius and $\widetilde{\Delta}$ is the spherical part of the Laplacian.

Next, we make the logarithmic change of variable and put

$$h(t,\theta) := e^{t/2} v(e^t x_\theta, e^t y_\theta, e^t z_\theta)$$

where $(x_\theta, y_\theta, z_\theta) = \theta \in \mathbb{S}^2$ and $t \in (-\infty, 0]$. Define $Q(t,\theta) = q(e^t x_\theta, e^t y_\theta)$. Then equation (4.5) turns into

$$\ddot{h} = \left(-\widetilde{\Delta} + e^{2t}(\gamma^2 - Q) + \frac{1}{4}\right) h =: L(t)h\,. \tag{4.6}$$

Note that $L(t)$ is a symmetric positive (unbounded) operator on the Hilbert space $L^2(\mathbb{S}^2)$. The initial conditions for ODE (4.6) are

$$\begin{aligned} h(-\infty) &= 0\,, \\ \dot{h}(-\infty) &= \lim_{r\to 0} r^{3/2} v_r = 0\,. \end{aligned}$$

Note that, due to our choice of γ, the derivative $\dot{L}(t) = 2e^{2t}\left(\gamma^2 - Q - \frac{1}{2}\dot{Q}\right)$ of $L(t)$ is also a non-negative operator.

At this point we make a break in the proof of the theorem and prove a lemma on second order ODEs (cf. Agmon [1]):

Lemma 4.7 *Let h be a solution to the equation*

$$\ddot{h} = L(t)h\,, \qquad -\infty < t \leqslant 0\,,$$

with

$$h(-\infty) = \dot{h}(-\infty) = 0$$

where $L(t)$ is a non-negative linear operator on a Hilbert space $\mathcal{H}$ such that $\dot{L}(t)$ is also non-negative. Then the function

$$t \mapsto \log \frac{\|h\|^2}{2}$$

is convex.

Proof: Denote $a(t) = \frac{1}{2}\|h\|^2$. Then $\dot{a}(t) = (h, \dot{h})$, and

$$\ddot{a}(t) = (h, \ddot{h}) + \|\dot{h}\|^2 = (h, L(t)h) + \|\dot{h}\|^2 \geqslant 0\,.$$

We need to show that $(\log a)^{\cdot\cdot} \geqslant 0$, or, equivalently, that $\ddot{a}a - \dot{a}^2 \geqslant 0$. We have

$$\begin{aligned} \ddot{a}a - \dot{a}^2 &= \left((L(t)h, h) + \|\dot{h}\|^2\right) \frac{\|h\|^2}{2} - (h, \dot{h})^2 \\ &\geqslant \left((L(t)h, h) + \|\dot{h}\|^2\right) \frac{\|h\|^2}{2} - \|h\|^2 \cdot \|\dot{h}\|^2 \\ &= \left((L(t)h, h) - \|\dot{h}\|^2\right) \frac{\|h\|^2}{2}\,. \end{aligned}$$

Further, since

$$\frac{d}{dt}\Big((L(t)h,h)-\|\dot h\|^2\Big)=(\dot L(t)h,h)+(L(t)\dot h,h)+(L(t)h,\dot h)-2(\ddot h,\dot h)=(\dot L(t)h,h),$$

we obtain

$$(L(t)h,h)-\|\dot h\|^2\geqslant\int_{-\infty}^{t}(\dot L(\tau)h,h)\,d\tau\geqslant 0\,,$$

and, thereby, $\ddot a a-\dot a^2\geqslant 0$, proving the lemma. □

Continuation of the proof of Theorem 4.3: Consider the spherical integral

$$I(t)=\frac{1}{2}\iint_{\mathbb{S}^2}h^2(t,\theta)\,d\sigma(\theta)$$

($d\sigma$ is the spherical area form) for the function h defined above. Since our function h is even in z-variable, we integrate only over the upper hemisphere $\mathbb{S}^2_+$. Introduce the coordinates

$$x=\rho\cos\varphi,\quad y=\rho\sin\varphi,\quad z=\sqrt{1-\rho^2}$$

on $\mathbb{S}^2_+$. Then

$$d\sigma=\frac{\rho}{\sqrt{1-\rho^2}}\,d\rho d\varphi\,,$$

and

$$h^2(t,\theta)=e^tF^2(e^t\rho,\varphi)\cosh^2(\gamma e^t\sqrt{1-\rho^2})\,.$$

We obtain

$$\begin{aligned}I(t)&=\int_0^{2\pi}d\varphi\int_0^1\frac{\rho\,d\rho}{\sqrt{1-\rho^2}}e^tF^2(e^t\rho,\varphi)\cosh^2(\gamma e^t\sqrt{1-\rho^2})\\&=\int_0^{e^t}\frac{s\,ds}{\sqrt{e^{2t}-s^2}}\cosh^2(\gamma\sqrt{e^{2t}-s^2})\int_0^{2\pi}F^2(s,\varphi)\,d\varphi\,.\end{aligned}$$

Finally, we introduce the function

$$J(r)=I(\log r)=\int_0^r\frac{\cosh^2(\gamma\sqrt{r^2-s^2})s}{\sqrt{r^2-s^2}}\left(\int_0^{2\pi}F^2(s,\varphi)\,d\varphi\right)ds\,,$$

By Lemma 4.7, the function $t\mapsto\log J(e^t)$ is convex. Hence

$$\frac{J(2s)}{J(s)}\leqslant\frac{J(2r)}{J(r)}\tag{4.8}$$

for $0 < s < r < \frac{1}{2}$.

It remains to rewrite this estimate in terms of $M(r)$. For this, we use the following standard lemma from the elliptic theory.

Lemma 4.9

$$c_3 e^{-\sqrt{N}r}\sqrt{\frac{J(r)}{r}} \leqslant M(r) \leqslant c_4 N\sqrt{\frac{J(2r)}{2r}}\,, \qquad 0 < r \leqslant \frac{1}{2}\,. \tag{4.10}$$

Proof of the upper bound: Observe that

$$M(r) \leqslant \frac{c_5 \max_{2r\mathbb{D}} |q|}{r} \left(\iint_{2r\mathbb{D}} F^2 \, d\,\text{Area} \right)^{1/2}. \tag{4.11}$$

Indeed, after rescaling, (4.11) reduces to its special case when $r = \frac{1}{2}$:

$$M(\tfrac{1}{2}) \leqslant c_6 \max_{\mathbb{D}} |q| \left(\iint_{\mathbb{D}} F^2 \, d\,\text{Area} \right)^{1/2}. \tag{4.12}$$

To get this estimate, we represent the function F as the sum of Green's potential and the Poisson integral:

$$F(z) = \iint_{\rho\mathbb{D}} q(\zeta)F(\zeta) \log\left|\frac{\rho^2 - z\overline{\zeta}}{\rho(z-\zeta)}\right| \, d\,\text{Area}(\zeta) + \int_{\rho\mathbb{T}} F(\zeta) \frac{\rho^2 - |z|^2}{|\zeta - z|^2} \, dm(\zeta)\,.$$

Here $|z| \leqslant \frac{1}{2}$, $\frac{2}{3} \leqslant \rho \leqslant 1$, and m is the normalized Lebesgue measure on the circle $\rho\mathbb{T}$. Then

$$M(\tfrac{1}{2}) \leqslant c_7 \left(\max_{\mathbb{D}} |q| \iint_{\rho\mathbb{D}} F^2 \, d\,\text{Area} + \int_{\rho\mathbb{T}} F^2 \, dm \right).$$

Averaging this by ρ over $[\frac{2}{3}; 1]$, we get (4.12).

By definition of the function J,

$$J(2r) \geqslant \frac{1}{2r} \iint_{2r\mathbb{D}} F^2 \, d\,\text{Area}\,,$$

and the upper bound in (4.10) follows from (4.11).

Proof of the lower bound:

$$\begin{aligned}
J(r) &\leqslant \int_0^r \frac{\cosh^2(\gamma\sqrt{r^2-s^2})}{\sqrt{r^2-s^2}} s\,ds \cdot 2\pi M^2(r) \\
&\leqslant r\cosh^2\gamma r \cdot \underbrace{\int_0^1 \frac{s\,ds}{\sqrt{1-s^2}}}_{=1} \cdot 2\pi M^2(r) \\
&\leqslant re^{2\gamma r}\cdot 2\pi M^2(r)\,.
\end{aligned}$$

It remains to recall that $\gamma = \sqrt{N}$. The lemma is proved. $\square$

End of the proof of Theorem 4.3: By Lemma 4.9,

$$\frac{M(2s)}{M(s)} \leqslant c_8 N e^{\sqrt{N}s} \cdot \left(\frac{J(4s)}{J(s)}\right)^{1/2},$$

$$\left(\frac{J(8r)}{J(2r)}\right)^{1/2} \leqslant c_9 N e^{8\sqrt{N}r} \cdot \frac{M(8r)}{M(r)},$$

and, by (4.8),

$$\left(\frac{J(4s)}{J(s)}\right)^{1/2} \leqslant \left(\frac{J(8r)}{J(2r)}\right)^{1/2}.$$

Juxtaposing these three inequalities, we obtain (4.4). $\square$

4.2 Proof of Theorem 4.1

Here and in the next section, we use the classical local description of smooth Riemannian metrics on closed surfaces. Given a point $p \in S$, one can choose local coordinates (x, y), $x^2 + y^2 \leqslant 1000$, near p so that the metric g in these coordinates is conformally Euclidean: $g = q(x, y)(dx^2 + dy^2)$. The point p corresponds to the origin: $p = (0, 0)$. This choice can be made in such a way that the function $q(x, y)$ is pinched between two positive constants that depend only on metric g:

$$0 < q_- \leqslant q(x, y) \leqslant q_+\,,$$

and that the C^1-norm of q is bounded by a constant depending only on the metric g.

Till the end of this section, by disc $D(p,r)$ centered at a point $p \in S$ with radius r we mean the set $\{x^2 + y^2 \leqslant r^2\}$, where (x,y) are local conformal coordinates near p. We can also choose our conformal charts in such way that for some $\eta > 0$ and any $p, p' \in S$ such that $\mathrm{dist}(p,p') < \eta$, we have $D(p, \frac{1}{2}) \subset D(p', 1)$.

We will refer to (x,y) as *preferred local conformal coordinates near* p. Note that in local conformal coordinates the eigenfunction $f := f_\lambda$ satisfies the equation

$$\Delta f + \lambda q(x,y) f = 0\,. \tag{4.13}$$

The proof of Theorem 4.1 is based on Theorem 4.3 and the following lemma, which, in turn, is also an easy consequence of Theorem 4.3 and compactness of the surface S.

Lemma 4.14 *For every point* $p \in S$,

$$\max_{D(p,1)} |f_\lambda| \geqslant e^{-a_2\sqrt{\lambda}} \max_S |f_\lambda|\,.$$

Proof of Lemma 4.14: Normalize the eigenfunction f_λ by the condition

$$\max_S |f_\lambda| = 1\,.$$

Let p_0 be the maximum point of $|f_\lambda|$ on S. For arbitrary $p \in S$, consider the chain of $k+1$ discs $D(p_j, 1)$ connecting p_0 with $p = p_k$ in such a way that

$$D(p_j, \tfrac{1}{2}) \subset D(p_{j+1}, 1)\,, \qquad 0 \leqslant j \leqslant k-1\,.$$

The number k depends only on the metric g. Due to Theorem 4.3 applied to solutions of equation (4.13) with $N \leqslant a_3\lambda$,

$$\frac{\max_{D(p_j,1)} |f_\lambda|}{\max_{D(p_j,\frac{1}{2})} |f_\lambda|} \leqslant e^{a_4\sqrt{\lambda}} \frac{\max_{D(p_j,8)} |f_\lambda|}{\max_{D(p_j,1)} |f_\lambda|} \leqslant \frac{e^{a_4\sqrt{\lambda}}}{\max_{D(p_j,1)} |f_\lambda|}\,,$$

or

$$\left(\max_{D(p_j,1)} |f_\lambda|\right)^2 e^{-a_4\sqrt{\lambda}} \leqslant \max_{D(p_j,\frac{1}{2})} |f_\lambda| \leqslant \max_{D(p_{j+1},1)} |f_\lambda|\,.$$

Making k iterations, we arrive at

$$e^{-a_2\sqrt{\lambda}} \leqslant \max_{D(p,1)} |f_\lambda|\,,$$

proving the lemma. $\square$

Proof of Theorem 4.1: Fix a disc $D(p,s) \subset S$ of radius s centered at p. Since in each conformal chart the Riemannian metric is equivalent to the Euclidean one: $q_-(dx^2+dy^2) \leqslant g \leqslant q_+(dx^2+dy^2)$, it suffices to show that

$$\frac{\max_{D(p,s)} |f_\lambda|}{\max_{D(p,\frac{1}{2}s)} |f_\lambda|} \leqslant e^{a_5\sqrt{\lambda}} \tag{4.15}$$

provided that $\lambda \geqslant 2$. The previous Lemma yields (4.15) for $s \geqslant 2$. Assume now that $s < 2$. We apply Theorem 4.3 to the solution f_λ of equation (4.13) in the disk $D(p,16)$. As above, the parameter N in Theorem 4.3 does not exceed $a_3\lambda$, and we immediately get (4.15). $\square$

5 Proof of Theorem 1.4

It suffices to prove Theorem 1.4 assuming that D is a Euclidean disc lying in a chart with preferred local coordinates and that the center of D belongs to the nodal line. All metric notions (distance, area and discs) pertain to the Euclidean metric.

We fix a positive number ρ_0 depending only on metric g such that

$$\rho_0^2 q_+ \leqslant \varepsilon_0 \tag{5.1}$$

where ε_0 is the numerical constant that controls 'smallness' of the potential in Section 3.

Definition 5.2 A *good* disc on S is a disc of radius $\leqslant \rho_0\lambda^{-1/2}$ whose center lies on the nodal line $\{f_\lambda = 0\}$.

Lemma 5.3 *Let D be a good disc. Then*

$$\frac{\mathrm{Area}(S_+(\lambda) \cap D)}{\mathrm{Area}(D)} \geqslant \frac{a_1}{\log\lambda \cdot \sqrt{\log\log\lambda}} \tag{5.4}$$

Proof of Lemma 5.3: Given a good disc D of radius $r\lambda^{-1/2}$, $r \in (0,\rho_0)$, with center p, define the function F on the unit disc $\mathbb{D} \subset \mathbb{C}$ by

$$F(x,y) = f_\lambda\left(\tfrac{rx}{\sqrt{\lambda}}, \tfrac{ry}{\sqrt{\lambda}}\right),$$

where (x,y) are preferred local coordinates near p. Then

$$\Delta F + r^2 q\left(\tfrac{rx}{\sqrt{\lambda}}, \tfrac{ry}{\sqrt{\lambda}}\right) F = 0\,,$$

and $F(0,0)=0$. We have

$$\frac{\mathrm{Area}(S_+(\lambda)\cap D)}{\mathrm{Area}(D)} \geqslant a_2\, \frac{\mathrm{Area}(\{F>0\})}{\mathrm{Area}(\mathbb{D})}\,.$$

Due to the choice of ρ_0, we can apply Theorem 3.2:

$$\frac{\mathrm{Area}(\{F>0\})}{\mathrm{Area}(\mathbb{D})} \geqslant \frac{a_3}{\log\beta^*(F)\cdot\sqrt{\log\log\beta^*(F)}}\,.$$

It follows from Theorem 4.1 that the right hand side is

$$\geqslant \frac{a_4}{\log\lambda\cdot\sqrt{\log\log\lambda}}\,.$$

This proves the lemma. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4: In the proof we use the fact that the inradius of every nodal domain does not exceed $\rho_1\lambda^{-1/2}$ where ρ_1 depends only on metric g (see [6]). Let D be a disc of radius R centered at the nodal line.

Case I: $R \geqslant 100\rho_1\lambda^{-1/2}$. Consider the collection of all discs of radii $2\rho_1\lambda^{-1/2}$ with centers on the nodal line $L=\{f_\lambda=0\}\cap\frac{1}{2}D$. Let $\{D_i\}_{i=1,\ldots,N}$ be a maximal subcollection of pairwise disjoint discs. We claim that every point $p\in\frac{1}{4}D$ lies at distance at most $6\rho_1\lambda^{-1/2}$ from the center of some D_i. Indeed, otherwise, choose a point $p'\in L$ with $\mathrm{dist}(p,p')\leqslant\rho_1\lambda^{-1/2}$, and consider the disc D' of radius $2\rho_1\lambda^{-1/2}$ centered at p'. Our assumption yields that D' is disjoint from all D_i's, which contradicts the maximality of the subcollection. The claim follows.

The claim yields that the discs $\{4D_i\}$ cover $\frac{1}{4}D$, so we get the inequality

$$\sum_i \mathrm{Area}(4D_i) \geqslant \mathrm{Area}(\tfrac{1}{4}D)\,. \tag{5.5}$$

Denote by D_i' the good disc $D_i' = \frac{\rho_0}{2\rho_1}\,D_i$. Note that $\mathrm{Area}(D_i')\geqslant a_5\,\mathrm{Area}(4D_i)$, and $\mathrm{Area}(\frac{1}{4}D)\geqslant a_6\,\mathrm{Area}(D)$. Therefore, using (5.5), we get

$$\sum_i \mathrm{Area}(D_i') \geqslant a_7\,\mathrm{Area}(D)\,. \tag{5.6}$$

Further,

$$\mathrm{Area}(S_+(\lambda)\cap D) \geqslant \sum_i \mathrm{Area}(S_+(\lambda)\cap D_i') \geqslant \frac{a_8}{\log\lambda\cdot\sqrt{\log\log\lambda}}\sum_i \mathrm{Area}(D_i')\,,$$

where the last inequality follows from Lemma 5.3. Combining this with (5.6), we obtain

$$\mathrm{Area}(S_+(\lambda)\cap D) \geqslant \frac{a_9}{\log\lambda\cdot\sqrt{\log\log\lambda}}\,\mathrm{Area}(D)\,,$$

which proves the theorem in this case.

Case II: $R \leqslant 100\rho_1\lambda^{-1/2}$. Choose ρ_0 in Definition 5.2 of good discs to be less than $400\rho_1$. Then the disc D' concentric with D of radius

$$r = R\cdot\frac{\rho_0}{400\rho_1}$$

is good. Applying Lemma 5.3, we get

$$\begin{aligned}\mathrm{Area}\,(S_+(\lambda)\cap D) &\geqslant \mathrm{Area}\,(S_+(\lambda)\cap D') \\ &\overset{(5.4)}{\geqslant} \frac{a_1\ \mathrm{Area}(D')}{\log\lambda\cdot\sqrt{\log\log\lambda}} \geqslant \frac{a_{10}\ \mathrm{Area}(D)}{\log\lambda\cdot\sqrt{\log\log\lambda}}\,.\end{aligned}$$

as required. This completes the proof in Case II, finishing off the proof of Theorem 1.4. □

6 Logarithmic asymmetry

In this section, we prove the results confirming sharpeness of our lower bounds for the area of positivity. First, we shall construct harmonic polynomials with small positivity area:

Theorem 6.1 *There exists a sequence of complex polynomials $P_N(z)$, $N = 2, 3, \ldots$, such that $\mathrm{deg}P_N = N$, $P_N(0) = 0$, and*

$$\mathrm{Area}(\{\mathrm{Re}\,P_N > 0\}\cap\mathbb{D}) \leqslant \frac{c}{\log N}\,.$$

Of course, Theorem 6.1 yields the upper bound for the Nadirashvili constant $\mathcal{N}$ in Theorem 1.8. Then we prove Theorem 1.5 'transplanting' the polynomials P_N to a small chart on the sphere $\mathbb{S}^2$ and transforming them into spherical harmonics on $\mathbb{S}^2$.

6.1 Proof of Theorem 6.1

Let us explain the idea behind the construction of harmonic polynomials in Theorem 6.1. We start with an entire function $E(z)$ on $\mathbb{C}$ which is bounded outside a semi-strip $\Pi_+ = \{x \geqslant 0,\ |y| \leqslant \frac{\pi}{2}\}$. For simplicity, assume that, for all sufficiently large R, the maximum $M(R) = \max_{R\mathbb{D}} |E|$ is attained at $z = R$ and $E(R)$ is real positive. Fix a sufficiently large R, and note that the function $G(z) := E(z+R) - E(R)$ vanishes at 0 and $\mathrm{Re}G < 0$ outside the strip $\Pi = \{|y| \leqslant \frac{\pi}{2}\}$. We will check that G admits a good approximation on the disc $R\mathbb{D}$ by its Taylor polynomial Q_N of degree $N \approx \log M(R)$, so the set $\{\mathrm{Re}Q_N > 0\} \cap R\mathbb{D}$ is still contained in the strip Π. Rescale the polynomial Q_N and set $P_N(z) := Q_N(Rz)$. Then

$$\begin{aligned}\frac{\mathrm{Area}(\{\mathrm{Re}P_N > 0\} \cap \mathbb{D})}{\mathrm{Area}(\mathbb{D})} &= \frac{\mathrm{Area}(\{\mathrm{Re}Q_N > 0\} \cap R\mathbb{D})}{\mathrm{Area}(R\mathbb{D})} \\ &\leqslant \frac{\mathrm{Area}(\Pi \cap R\mathbb{D})}{\mathrm{Area}(R\mathbb{D})} \leqslant \frac{c_1}{R} \approx \frac{c_2}{M^{-1}(e^N)} \qquad (6.2)\end{aligned}$$

where M^{-1} is the inverse function to the function M.

To get the optimal example, we have to minimize the right hand side of (6.2), that is, to start with the function E as above with the minimal possible growth. According to the Phragmén-Lindelöf principle, the minimal growth rate for $M(R)$ is of the double exponent order $\exp\exp R$. Therefore, $N \approx \exp R$, and (6.2) yields

$$\frac{\mathrm{Area}(\{\mathrm{Re}P_N > 0\} \cap \mathbb{D})}{\mathrm{Area}(\mathbb{D})} \leqslant \frac{c_3}{\log N}\,,$$

as needed.

Now, let us pass to the formal construction.

Proof of Theorem 6.1: Following Mittag-Leffler and Malmquist, we produce an entire function $E(z)$ such that

$$|E(z)| \leqslant c_4 \qquad \text{for} \quad z \notin \Pi_+\,, \qquad (6.3)$$

and

$$\left|E(z) - e^{e^z}\right| \leqslant c_4 \qquad \text{for} \quad z \in \Pi_+\,. \qquad (6.4)$$

To get E, denote $\Pi' = \{x > 0,\ |y| \leqslant \frac{2}{3}\pi\}$, $\Pi'' = \{x > -1,\ |y| \leqslant \frac{4}{3}\pi\}$, and consider a smooth cut-off function χ on $\mathbb{C}$ that equals 1 on Π' and vanishes

outside Π''. In addition, choose χ so that $\left|\bar\partial\chi\right|$ is uniformly bounded. Define

$$u(z)=\frac{1}{\pi}\iint_{\mathbb{C}}\frac{e^{e^\zeta}\bar\partial\chi(\zeta)}{z-\zeta}\,d\,\mathrm{Area}(\zeta)=\frac{1}{\pi}\iint_{\Pi''\setminus\Pi'}\frac{e^{e^\zeta}\bar\partial\chi(\zeta)}{z-\zeta}\,d\,\mathrm{Area}(\zeta)\,. \qquad (6.5)$$

Then

$$\bar\partial u=e^{e^z}\bar\partial\chi=\bar\partial\left(\chi e^{e^z}\right)\,,$$

so the function $E(z)=\chi\exp\exp z-u(z)$ is entire. To establish properties (6.3) and (6.4), it suffices to show that $|u|$ is bounded. This readily follows from the fact that $|\exp\exp z|=\exp(e^x\cos y)$, and, therefore, the integrand in (6.5) decays very rapidly when $\zeta\to\infty$ within the layer $\Pi''\setminus\Pi'$.

Now choose $R>0$ such that $R>2c_4+1$, and set $G(z)=E(z+R)-E(R)$. Then $G(0)=0$, and, for $z\notin\Pi:=\{|\operatorname{Im}z|\leqslant\frac{\pi}{2}\}$,

$$\mathrm{Re}G(z)\leqslant c_4-(e^{e^R}-c_4)\leqslant-1\,.$$

If r is sufficiently large, we have

$$M(r):=\max_{r\mathbb{D}}|G(z)|\leqslant\exp(c_5e^r)\,.$$

It remains to approximate G by its Taylor polynomial[3]. Let

$$G(z)=\sum_{n=1}^{\infty}a_nz^n\,,\qquad Q_N(z)=\sum_{n=1}^{N}a_nz^n\,,\qquad R_N(z)=G(z)-Q_N(z)\,.$$

By Cauchy's inequalities,

$$|a_n|\leqslant\frac{M(\rho)}{\rho^n}\leqslant\exp(c_5e^\rho-n\log\rho)\,.$$

Assume that n is sufficiently large, and choose ρ so that $c_5\rho e^\rho=n$. Then

$$\frac{n}{c_5\log n}\leqslant e^\rho\leqslant n\,,$$

and

$$|a_n|\leqslant\exp\left(c_5n-n\log\log\frac{n}{c_5\log n}\right)\leqslant\left(\frac{c_6}{\log n}\right)^n\,.$$

[3]This step is not needed for the upper bound for the Nadirashvili constant $\mathcal{N}$ in Theorem 1.8 that can be obtained directly by scaling $\mathrm{Re}G$.

Therefore, for $|z| = r$,

$$|R_N(z)| \leqslant \sum_{n=N+1}^{\infty} |a_n| r^n \leqslant \sum_{n=N+1}^{\infty} \left(\frac{c_6 r}{\log n} \right)^n .$$

Hence, if $r \leqslant r_N := \frac{1}{2} c_6^{-1} \log N$ and N is large enough, we have $|R_N(z)| \leqslant \frac{1}{2}$, which yields $\mathrm{Re} Q_N(z) \leqslant -\frac{1}{2}$ for $|z| \leqslant r_N$, $|\mathrm{Im} z| \geqslant \frac{\pi}{2}$. Finally, make a rescaling $P_N(z) = Q_N(r_N z)$. This is a polynomial of degree N with $P_N(0) = Q_N(0) = 0$, and

$$\begin{aligned} \frac{\mathrm{Area}(\{\mathrm{Re} P_N > -\frac{1}{2}\} \cap \mathbb{D})}{\mathrm{Area}(\mathbb{D})} &= \frac{\mathrm{Area}(\{\mathrm{Re} Q_N > -\frac{1}{2}\} \cap r_N \mathbb{D})}{\mathrm{Area}(r_N \mathbb{D})} \\ &\leqslant \frac{\mathrm{Area}(\Pi \cap r_N \mathbb{D})}{\mathrm{Area}(r_N \mathbb{D})} \leqslant \frac{2\pi r_N}{\pi r_N^2} = \frac{2}{r_N} = \frac{2 c_6}{\log N} . \end{aligned}$$

This completes the proof of Theorem 6.1. □

6.2 Proof of Theorem 1.5

We work on the sphere $\mathbb{S}^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ endowed with the standard spherical metric. The spectrum of the Laplacian on $\mathbb{S}^2$ is given by $\lambda_N = N(N+1)$, where each λ_N has multiplicity $2N+1$. Put $A = (0,0,1)$, and consider the upper hemi-sphere $\mathbb{S}^2_+ = \{x_3 > 0\}$. Then (x_1, x_2) are local coordinates on $\mathbb{S}^2_+$. Put $z = x_1 + i x_2$, $r = \sqrt{x_1^2 + x_2^2}$, $z = r e^{i\theta}$. Consider the space $\mathcal{S}_N$ of complex valued spherical harmonics corresponding to the eigenvalue λ_N that vanish at A. Clearly, $\dim_{\mathbb{C}} \mathcal{S}_N = 2N$. We shall use the following classical

Lemma 6.6 *There exists a basis e_1, e_2, ..., e_N, e_{-1}, e_{-2}, ..., e_{-N} in $\mathcal{S}_N$ such that each function e_j restricted to $\mathbb{S}^2_+$ has the form*

$$e_j(x_1, x_2) = L_N^{(j)}(\sqrt{1 - r^2}) z^j , \qquad j = 1, ..., N ,$$

$$e_{-j}(x_1, x_2) = L_N^{(j)}(\sqrt{1 - r^2}) \bar{z}^j , \qquad j = 1, ..., N .$$

Here L_N is the Legendre polynomial of degree N, and $L_N^{(j)}$ stands for its j-th derivative.

For the proof, see, e.g., [13, Lemma 3.5.3].

Proof of Theorem 1.5: Write $L_N^{(j)}(\sqrt{1-r^2}) = A_{jN} + rB_{jN}(r)$, where B_{jN} is a continuous function on $[0;1]$, and $A_{jN} = L_N^{(j)}(1)$. Since all zeroes of the Legendre polynomial L_N are real and lie in the interval $(-1;1)$, and since its leading coefficient is positive, we have $A_{jN} > 0$. In view of Theorem 6.1, there exist a sequence of complex polynomials $P_N(z) = \sum_{j=1}^{N} \alpha_{jN} z^j$ and a sequence of small positive values $\{\kappa_N\}$ such that

$$\text{Area}(\{z \in \mathbb{D}\colon \text{Re}P_N(z) > -\kappa_N\}) \leqslant \frac{c_7}{\log N}\,. \tag{6.7}$$

Let $\delta = \delta_N$ be a sufficiently small positive number (to be chosen later). Fix N large enough, and consider the spherical harmonic $f_N \in \mathcal{S}_N$ defined by,

$$f_N(r,\theta) = \sum_{j=1}^{N} \beta_j L_N^{(j)}(\sqrt{1-r^2}) r^j e^{ij\theta}$$

with

$$\beta_j = \frac{\alpha_j}{A_j \delta^j}$$

(to simplify notation, we suppress the subindex N for the coefficients α_j, β_j and A_j, as well as for the functions B_j). Rescaling, define

$$F_N(r,\theta) = f_N(\delta r,\theta) = \sum_{j=1}^{N} \beta_j \left(A_j + \delta r B_j(\delta r)\right) \delta^j r^j e^{ij\theta}\,.$$

Then

$$\left|F_N(r,\theta) - P_N\left(re^{i\theta}\right)\right| = \left|\sum_{j=1}^{N} \frac{\alpha_j \delta r B_j(\delta r)}{A_j} r^j e^{ij\theta}\right| \leqslant \delta M_N$$

where

$$M_N = \max_{1\leqslant j\leqslant N} \max_{r\in[0;1]} \left|\frac{B_j(r)}{A_j}\right| \cdot \sum_{j=1}^{N} |\alpha_j|\,.$$

Choose $\delta = \delta_N$ so small that

$$\delta M_N < \kappa_N\,.$$

Put $E_N = \{|z| < \delta_N\}$, $D_N = \{(x_1, x_2, x_3) \in \mathbb{S}^2 \colon x_1 + ix_2 \in E_N\}$. We have (writing Area_e and Area_s for the euclidean and the spherical areas respectively)

$$\begin{aligned}\frac{\mathrm{Area}_e(\{\mathrm{Re} f_N > 0\} \cap E_N)}{\mathrm{Area}_e(E_N)} &= \frac{\mathrm{Area}_e(\{\mathrm{Re} F_N > 0\} \cap \mathbb{D})}{\mathrm{Area}_e(\mathbb{D})} \\ &\leqslant \frac{\mathrm{Area}_e(\{\mathrm{Re} P_N > -\kappa_N\} \cap \mathbb{D})}{\mathrm{Area}_e(\mathbb{D})} \leqslant \frac{c_7}{\pi \log N}\,.\end{aligned}$$

At the same time,

$$\frac{\mathrm{Area}_e(\{\mathrm{Re} f_N > 0\} \cap E_N)}{\mathrm{Area}_e(E_N)} \geqslant c_8 \cdot \frac{\mathrm{Area}_s(\{\mathrm{Re} f_N > 0\} \cap D_N)}{\mathrm{Area}_s(D_N)}\,.$$

This yields

$$\frac{\mathrm{Area}_s(\{\mathrm{Re} f_N > 0\} \cap D_N)}{\mathrm{Area}_s(D_N)} \leqslant \frac{c_7 \cdot c_8^{-1}}{\pi \log N}$$

for all N, as required. □

7 Discussion and questions

7.1 Quasi-conformal or C^1-smooth?

The link between harmonic functions and Laplace-Beltrami eigenfunctions on surfaces given in Lemma 3.4 above plays a crucial role in the present paper. Recall that the lemma states that, for any solution F to the Schrödinger equation $\Delta F + qF = 0$ in the disc $\mathbb{D}$ with small smooth potential q, there exist a harmonic function $U \colon \mathbb{D} \to \mathbb{R}$, a positive function φ, and a quasi-conformal homeomorphism h of $\mathbb{D}$ such that $F = \varphi \cdot (U \circ h)$. It remains unclear to us whether h can be chosen to be C^1-smooth (or Lipschitz) with controlled differential:

$$\|dh\|, \|dh^{-1}\| \leqslant C(\|q\|).$$

Such a result would immediately remove the double logarithm in the area estimates presented in Theorems 1.4 and 3.2. The refined estimates would be sharp in view of Theorem 1.5.

7.2 At which scale does quasisymmetry break?

Recall that Theorem 1.5 establishes the existence of a sequence of spherical harmonics $\{f_i\}$ on the 2-sphere corresponding to eigenvalues $\lambda_i \to \infty$ and a sequence of discs $D_i \subset \mathbb{S}^2$ such that each f_i vanishes at the center of D_i and

$$\frac{\mathrm{Area}(S_+(\lambda_i) \cap D_i)}{\mathrm{Area}(D_i)} \leqslant \frac{C}{\log \lambda_i} .$$

In our proof, the radii r_i of the discs D_i decay very rapidly as the eigenvalues λ_i tend to infinity. It would be interesting to explore what is the optimal (that is, the slowest) possible rate of decay of the r_i's for which the inequality above is still valid. For instance, can this happen on the wave-length scale $r_i \sim 1/\sqrt{\lambda_i}$? Let us emphasize that the sequence r_i must converge to zero in view of the fact that spherical harmonics enjoy quasisymmetry

$$\frac{\mathrm{Area}(S_+(\lambda_i) \cap D)}{\mathrm{Area}(D)} \geqslant \mathrm{const}(r)$$

for any disc D of radius $\geqslant r$ (this follows from [9] and [23] since the spherical metric is real analytic).

7.3 The doubling exponent: from uniform measurements to statistics

The next discussion is a result of our attempt to digest Nadirashvili's approach in [23]. Start with a sequence of eigenfunctions f_λ, $\lambda \to +\infty$. Fix $r > 0$ small enough, and consider the function

$$b(x, \lambda) := \beta(D(x, \tfrac{r}{\sqrt{\lambda}}), f_\lambda)$$

where $D(x, \frac{r}{\sqrt{\lambda}})$ stands for the metric disc of radius $\frac{r}{\sqrt{\lambda}}$ with the center at the point $x \in S$. Recall the Donnelli-Fefferman estimate

$$B_\infty(\lambda) := \sup_{x \in S} b(x, \lambda) \leqslant c\sqrt{\lambda},$$

which played a crucial role in our approach. Interestingly enough, replacing the L_∞-norm of b by the L_1- norm

$$B_1(\lambda) := \frac{1}{\mathrm{Area}(S)} \int_S b(x, \lambda) d\,\mathrm{Area}(x)\,,$$

we get a quantity that is closely related to the length of the nodal line $L_\lambda := \{f_\lambda = 0\}$:

$$C^{-1} \cdot \mathrm{Length}(L_\lambda)\lambda^{-1/2} - C \leqslant B_1(\lambda) \leqslant C \cdot \mathrm{Length}(L_\lambda)\lambda^{-1/2} + C, \qquad (7.1)$$

where the constant $C > 1$ depends only on metric g on the surface S.

Here is a sketch of the proof. Define $N(x,\lambda)$ to be the number of intersection points between the boundary circle T of the disc $D := D(x, \frac{r}{\sqrt{\lambda}})$, and the nodal line L_λ. Let U be the harmonic function associated to $f_\lambda|_D$ as in Lemma 3.4. Then $N(x,\lambda)$ equals the number of sign changes of U on $\partial\mathbb{D}$. Using the topological interpretation of the doubling exponent of harmonic functions (see formula (1.7)) it is possible to show that

$$N(x,\lambda) \simeq \beta(\mathbb{D}, U) \simeq \beta(D, f_\lambda) = b(x,\lambda).$$

Applying an elementary integral geometry argument we get

$$\mathrm{Length}(L_\lambda) \simeq \int_S N(x,\lambda) d\,\mathrm{Area}(x) \cdot \sqrt{\lambda} \simeq B_1(\lambda)\sqrt{\lambda},$$

which readily yields inequality (7.1).

Inequality (7.1) clarifies the function-theoretic meaning of the Yau conjecture for surfaces, which states that $\mathrm{Length}(L_\lambda) \simeq \sqrt{\lambda}$: *the expectation of the doubling exponent of f_λ on a random metric disc of radius $\sim 1/\sqrt{\lambda}$ is bounded by a constant depending only on the Riemannian metric.* The Yau conjecture was proved in [9] in any dimension for real analytic metrics g.

7.4 What happens in higher dimensions?

It would be interesting to extend Theorem 1.4 and to explore the local asymmetry of the sign distribution for eigenfunctions on higher-dimensional manifolds. This problem has the following counterpart for harmonic functions on the unit ball $\mathbb{B} \subset \mathbb{R}^n$, $n \geqslant 3$. Let u be a non-zero harmonic function on $\mathbb{B}$ vanishing at the origin. What is the optimal bound for $\mathrm{Vol}(\{u > 0\})$ in terms of its doubling exponent $\beta(\mathbb{B}, u)$? Using Carleman's method [7] or otherwise, one can easily show that

$$\mathrm{Vol}(\{u > 0\}) \geqslant \frac{c}{(\beta(\mathbb{B}, u))^{n-1}}\,.$$

However, we believe that this estimate is very far from being sharp.

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Fëdor Nazarov
Department of Mathematics
Michigan State University
East Lansing, MI 48824
USA
fedja@math.msu.edu

Leonid Polterovich
School of Mathematics
Tel Aviv University
Tel Aviv 69978
Israel
polterov@post.tau.ac.il

Mikhail Sodin
School of Mathematics
Tel Aviv University
Tel Aviv 69978
Israel
sodin@post.tau.ac.il